$\mathrm{Ind}(\mathrm{FPFlat}_R) < \mathrm{Flat}_R$ for coherent non-regular R

Let FPFlat_R denote the category of finitely presented flat commutative R-algebras, and let Flat_R denote the category of flat commutative R-algebras. There is a natural functor

$$\Psi : \mathrm{Ind}(\mathrm{FPFlat}_R) \longrightarrow \mathrm{Flat}_R$$

given by taking the filtered colimit in Alg_R. This note shows that Ψ is not essentially surjective for any coherent ring R which is not regular. The result provides more examples of a phenomenon discussed in [1] and [2] (see also Tag 0ATE in the Stacks Project [3]).

Theorem 1.

$$\Psi : \mathrm{Ind}(\mathrm{FPFlat}_R) \longrightarrow \mathrm{Flat}_R$$

is not essentially surjective for any coherent ring R which is not regular.

Proof. [1]. All modules and tensor products are implicitly derived. First suppose $(R, \mathfrak{m}, k)$ is local. Let

$$S := R \oplus Rt \oplus \bigoplus_{i \geq 0} F_i^\vee,$$

where $F_\bullet$ is a minimal resolution of a finitely generated ideal I of infinite projective dimension. The multiplication is defined so that $t^2 = 0$, the ideal $\oplus F_i^\vee$ is square-zero, and t acts as the dual of the differential on $\oplus F_i^\vee$. Define η to be the $R[t]/t^2$-linear map

$$R \longrightarrow I \otimes_R S$$

sending 1_R to the element in $I \otimes_R F_0^\vee$ corresponding to the augmentation map. This map can be represented by the map of chain complexes $(\dots R[t]/t^2 \xrightarrow{t} R[t]/t^2) \to (\dots F_1 \otimes_R S \to F_0 \otimes_R S)$ where all the chain maps correspond to the distinguished element in $F_i \otimes_R F_i^\vee$ corresponding to the identity map.

We claim this map can never factor through $I \otimes_R B \to I \otimes_R S$ for map of $R[t]/t^2$-algebra $B \to S$ where B is finitely generated over $R[t]/t^2$. Indeed, after base-changing to $k[t]/t^2$, the map

$$k \longrightarrow I \otimes_R S \otimes_R k \cong I \otimes_R \left(k \cdot 1 \oplus k \cdot t \oplus \bigoplus_{i \geq 0} (F_i^\vee \otimes_R k) \right)$$

hits infinitely many components of the decomposition, as can be seen by the chain complex model. On the other hand, because the $F_i^\vee$'s in S are square-zero, the image of $B \otimes_R k \to S \otimes_R k$ is contained in a finite-dimensional $k[t]/t^2$-submodule of $S \otimes_R k$, and hence misses $F_i^\vee \otimes_R k$ for all $i \gg 0$.

If S were a filtered colimit of R-flat finitely generated R-algebras, then, after passing to a cofinal system, it would be a filtered colimit of $R[t]/t^2$-algebras flat over R, and the map $R \to I \otimes_R S$ would

[1] a variant of this argument was first discovered by GPT 5

factor through some such $I \otimes_R B$. The obstruction above rules this out. In fact, the same argument shows that S also cannot be written as a filtered colimit of finitely generated E_∞-R-algebras with uniformly bounded Tor-amplitude over R.

Finally, a coherent ring R which is non-regular has a localization at a prime which is non-regular. Hence we deduce the result in general. $\square$

References

[1] Bhargav Bhatt, *A flat map that is not a directed limit of finitely presented flat maps*, available at `https://www.math.ias.edu/~bhatt/math/flat-maps-limits-fp.pdf`.

[2] O. Gabber, *A property of non excellent rings*, manuscripta mathematica, 91(1):543–546, 1996. DOI: 10.1007/BF02567973.

[3] The Stacks Project Authors, *The Stacks Project*, Tag 0ATE, `https://stacks.math.columbia.edu/tag/0ATE`.