ALGEBRAIZATION OF ABELIAN TORSORS

Abstract. We give an alternative proof, based on Antieau-Gepner Algebraization results in [AG14] (following work of To¨en-Vezzosi [TV07]), of Theorem 4.3 in [RR25]

$$H^1(\mathcal{A}) \cong \mathrm{Ext}^2(\mathcal{A}^\vee, \mathbb{G}_m)$$

for an abelian scheme $\mathcal{A}$ over S, where S is allowed to be any spectral affine scheme. From this, we deduce using the Breen-Deligne resolution that $(B\mathcal{A})^{\#}_{\text{ét}}$ has descent along any cover $S' \to S$ such that $(\mathcal{A}^\vee)^n \times_S S' \to (\mathcal{A}^\vee)^n$ has descent for $(B^2\mathbb{G}_m)^{\#}_{\text{ét}}$ for all n.

Fix a base spectral affine scheme $S := \mathrm{Spec}\, R$ and let $p_{\mathcal{A}} : \mathcal{A} \to S$ be an abelian scheme over S in the sense of Lurie. [RR25] Theorem 4.3 follows from sections of $\mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(\_,\mathrm{Sp}^{\mathrm{cn}})}(\mathcal{A}^\vee, \mathbb{G}_m[2])$ being étale locally trivial. To show this, we will algebraize the space of trivializations of a given section, and show that this algebraic stack is smooth and has sections over geometric points. The basic technical input for this algebraization is

Theorem 0.1 (Antieau-Gepner)**.** *Let $S := \mathrm{Spec}\, R$ where R is a connective $\mathbb{E}_\infty$-ring. Let C be a smooth and proper category over* $\mathrm{Perf(S)}$*. Then, the moduli of objects in C is a quasi-separated, locally geometric, and locally of finite presentation stack (see [AG14] Section 4.3 for the definitions).*

Remark 0.2. *We note that, to the incomplete knowledge of the author, the algebraicity results of [AG14] (following [TV07]) constitute one of a very limited number of ways to prove algebraicity without Artin's criterions–which we don't currently have for higher stacks in spectral algebraic geometry. The power of this approach is seen by the fact that it instantly implies the algebraizability of Picard stacks for smooth proper morphisms.*

The main algebraization result we need follows from (we will only use that $\mathcal{A}^\vee \to S$ is a $\mathbb{E}_1$-smooth and proper morphism)

Proposition 0.3. *The map of sheaves on the big étale site of S*

$$* \xrightarrow{0} p^{\acute{e}t}_{\mathcal{A}^\vee,*}(B^2\mathbb{G}_m) \cong \mathrm{Hom}_{\mathrm{Shv}^{\acute{e}t}(\_,\mathcal{S})}(\mathcal{A}^\vee, B^2\mathbb{G}_m)$$

is representable by a quasi-separated n-geometric stack locally of finite presentation for some finite n.

Proof. Let X be an affine scheme over S and u be a map $\mathcal{A}^\vee_X \to (B^2\mathbb{G}_m)^{\#}_{\text{ét}}$. Then, u determines a compactly generated invertible $\mathrm{QCoh}(\mathcal{A}^\vee_{\mathrm{X}})^{\mathrm{cn}}$-module category C_u (in Groth_∞)[1]. The space of trivializations of u (equivalently of C_u)

$$X \times_{p^{\text{ét}}_{\mathcal{A}^\vee,*}(B^2\mathbb{G}_m)} S \tag{0.1}$$

is a subsheaf[2] of the moduli of objects in C_u (whose value on an affine X-scheme Y is the space of objects in $C_u \otimes_{\mathrm{QCoh}(\mathcal{A}^\vee_{\mathrm{X}})^{\mathrm{cn}}} \mathrm{QCoh}(\mathcal{A}^\vee_{\mathrm{Y}})^{\mathrm{cn}} \cong C_u \otimes_{\mathrm{QCoh(X)}^{\mathrm{cn}}} \mathrm{QCoh(Y)}^{\mathrm{cn}}$).[3] In fact it lands inside the moduli of compact[4] and flat objects (meaning the map $\mathrm{QCoh}(\mathcal{A}^\vee_{\mathrm{X}})^{\mathrm{cn}} \to C_u$ determined by acting on the object preserves finite limits).

[1] see Theorem D.4.1.2 and Theorem D.5.3.1 of [Lur18]
[2] meaning the inclusion is (-1)-truncated
[3] see Proposition C.3.2.4 and Proposition 10.2.2.3 in [Lur18]
[4] see [Lur18] Remark D.5.2.3

We claim that the inclusion

$$X \times_{p^{\text{ét}}_{\mathcal{A}^\vee,*}(B^2\mathbb{G}_m)} S \to \text{Moduli}_{/X}(\text{Sp}(C_u)^\omega)$$

is a quasi-compact open immersion. The proof of Proposition 5.10 in [AG14] shows that

$$X \times_{p^{\text{ét}}_{\mathcal{A}^\vee,*}(B\text{Pic}^{\text{Perf}})} S \to \text{Moduli}_{/X}(\text{Sp}(C_u)^\omega)$$

is a quasi-compact open immersion (i.e. the space of trivializations of the $\text{Perf}(\mathcal{A}^\vee_{\text{X}})$-invertible category $\text{Sp}(C_u)^\omega$ is quasi-compact open in the moduli of objects therein). So it suffices for our claim that

$$X \times_{p^{\text{ét}}_{\mathcal{A}^\vee,*}(B^2\mathbb{G}_m)} S \to X \times_{p^{\text{ét}}_{\mathcal{A}^\vee,*}(B\text{Pic}^{\text{Perf}})} S \cong X \times_{p^{\text{ét}}_{\mathcal{A}^\vee,*}(B^2\mathbb{G}_m)} p^{\text{ét}}_{\mathcal{A}^\vee,*}\mathbb{Z}_{\mathcal{A}^\vee} \cong X \times_{p^{\text{ét}}_{\mathcal{A}^\vee,*}(B^2\mathbb{G}_m)} \mathbb{Z}_S$$

is a quasi-compact open immersion, which is clear.

As $\text{Perf}(\mathcal{A}^\vee_{\text{X}})$ is a smooth and proper Perf(X)-category, thus so is $\text{Sp}(C_u)^\omega$ (as all dualizable modules of dualizable algebras are dualizable). So the moduli of objects is a quasi-separated locally geometric stack which is locally of finite presentation by Theorem 0.1 above.

However, to obtain that the morphism $* \xrightarrow{0} p^{\text{ét}}_{\mathcal{A}^\vee,*}(B^2\mathbb{G}_m)$ is n-geometric for some n, we require a slightly finer analysis. Choosing a generator $F \in C_u$ and shifting appropriately, we can find a conservative functor

$$\text{Sp}(C_u)^\omega \to \text{Perf(X)}$$

which sends flat objects to objects with tor-amplitude in $[a, b]$ for some fixed a and b. [AG14] Proposition 5.6 and Proposition 5.7 implies that the subsheaf of $\text{Moduli}_{/X}(\text{Sp}(C_u)^\omega)$ whose image under the functor has tor-amplitude in $[a, b]$ is n-geometric for some n. As (0.1) factors through this subsheaf, it is also n-geometric. ■

Proposition 0.4. *Let u be a point in $\text{Hom}_{\text{Shv}^{ét}(S,\text{Sp}^{\text{cn}})}(\mathcal{A}^\vee, \mathbb{G}_m[2])$, then the space of homotopies between u and 0, which we denote $\text{Eq}(u, 0)$, is representable by a smooth n-geometric stack where $n = 1 + \dim(A/S)$.*

Proof. By the spectral Breen-Deligne resolution (in the form of [Aok24] Theorem 4.4[5]), we have the isomorphism

$$\text{Eq}(u, 0) \cong \lim_{[i]\in\mathbf{\Delta}}(\text{Eq}(u^{(i)}, 0))$$

where $u^{(i)}$ is a section of

$$\text{Hom}_{\text{Shv}^{\text{ét}}(\_,S)}(\mathcal{A}^{n_i}, B^2\mathbb{G}_m)$$

and the n_i's are nonnegative integers whose values are not relevant for us. Proposition 0.3 and Lemma 4.36 in [AG14] shows that $\text{Eq}(u, 0)$ is representable by a n-geometric stack locally of finite presentation. Smoothness can be calculated with a cotangent complex calculation (the cotangent complex in general agrees with the case $u = 0$, where $\text{Eq}(u, 0) \cong \mathcal{A}$). ■

Remark 0.5. *The case $u = 0$ in Proposition 0.3 gives an independent proof of the algebraicity of the relative Picard stack of a smooth and proper morphism. The case $u = 0$ in Proposition 0.4 gives an independent proof of the algebraicity of the dual abelian scheme.*

We wish to show that any section of

$$\text{Hom}_{\text{Shv}^{\text{ét}}(\_,\text{Sp}^{\text{cn}})}(\mathcal{A}, \mathbb{G}_m[2])$$

vanishes étale locally. Because the space of trivializations is representable by a smooth algebraic stack, it suffices to show it on geometric points. The following is a (very mild) generalization of a result of [Bre69].

Proposition 0.6. *If S is the spectrum of an algebraically closed field, then*

$$\mathcal{A}^\vee[1] \cong \text{Hom}_{\text{Shv}^{ét}(\_,\text{Sp}^{\text{cn}})}(\mathcal{A}, \mathbb{G}_m[2])$$

[5] see also [MMY25] Lemma 2.4.3

Proof. We emulate the proof of Lemma 4.4 in [RR25]. Duality theory of abelian schemes implies that we just need to show that

$$\mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(k,\mathrm{Sp}^{\mathrm{cn}})}(\mathcal{A}, \mathbb{G}_m[2])$$

is a connected homotopy type. This follows from showing that π_0 is torsion and

$$\mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(k,\mathrm{Sp}^{\mathrm{cn}})}(\mathcal{A}[n], \mathbb{G}_m[1])$$

is connected (which implies multiplication by n on π_0 is injective). The latter is SGA 7I, Exp. VIII, Prop. 3.3.1. The former is (almost) the computation in [Bre69] except we are doing the computation over the sphere instead of over $\mathbb{Z}$–but in fact this doesn't matter as we see below.

To show that

$$\pi_k \,\mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(k,\mathrm{Sp})}(\mathcal{A}, \mathbb{G}_m)$$

is torsion except for $k = -1$, it suffices to show that

$$\pi_k \,\mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(k,\mathrm{Sp})}(\mathrm{Hom}_{\mathbb{S}}(\mathbb{Q}, \mathcal{A}), \mathbb{G}_m)$$

is zero except for $k = -1$. This is because of the following. First, $\mathrm{Hom}_{\mathbb{S}}(\mathbb{Q}, \mathcal{A})$ (as a sheaf of spaces) is representable by a ind-(finite flat) cover of $\mathcal{A}$. Now, the étale cohomology of smooth group schemes commutes with filtered colimits of discrete rings by SGA 4II, Exp. VII, Cor. 5.9. Hence we have, by using the truncated Breen-Deligne resolution (which is allowed because the rings are discrete)

$$\pi_k \,\mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(k,\mathrm{Sp})}(\mathrm{Hom}_{\mathbb{S}}(\mathbb{Q}, \mathcal{A}), \mathbb{G}_m) \cong \mathbb{Q} \otimes \pi_k \,\mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(k,\mathrm{Sp})}(\mathcal{A}, \mathbb{G}_m)$$

But we have

$$\mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(k,\mathrm{Sp})}(\mathrm{Hom}_{\mathbb{S}}(\mathbb{Q}, \mathcal{A}), \mathbb{G}_m) \cong \mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(k,\mathrm{Mod}\,\mathbb{Z})}(\mathbb{Z} \otimes \mathrm{Hom}_{\mathbb{S}}(\mathbb{Q}, \mathcal{A}), \mathbb{G}_m)$$

which is also

$$\mathbb{Z} \otimes \pi_k \,\mathrm{Hom}_{\mathrm{Shv}^{\text{ét}}(k,\mathrm{Mod}\,\mathbb{Z})}(\mathcal{A}, \mathbb{G}_m)$$

hence the result follows from [Bre69]

■

Corollary 0.7. *The natural map of presheaves of connective spectra (on affine schemes over S–which we denote* $\mathrm{Aff}_{/S}$*)*

$$\mathcal{A}^\vee[1] \to \mathrm{Hom}_{\mathrm{Shv}^{\acute{e}t}(\_,\mathrm{Sp}^{\mathrm{cn}})}(\mathcal{A}, \mathbb{G}_m[2])$$

is an isomorphism upon étale sheafification. Hence $(B\mathcal{A})^{\#}_{\acute{e}t}$ *has descent along any map* $S' \to S$ *such that* $(B^2\mathbb{G}_m)^{\#}_{\acute{e}t}$ *has descent on* $(\mathcal{A}^\vee)^n \times_S S' \to (\mathcal{A}^\vee)^n$ *for all* $n \geq 0$*.*

Proof. The map is (-1)-truncated so it suffices to show every section of the right hand side is étale-locally trivial. But the space of trivializations (of a section u) is a smooth n-geometric stack by Proposition 0.4 which has sections on geometric points (by Proposition 0.6). Hence it has sections étale-locally by Theorem 4.47 in [AG14]. The last part follows from the Breen-Deligne resolution. ■

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