On Splinterites

Theorem. Let k be a field and S a finite-dimensional k-algebra. Put $S^e := S \otimes_k S^{\mathrm{op}}$. For a left S-module V with action $\rho : S \to \mathrm{End}_k(V)$, give $\mathrm{End}_k(V)$ the S-bimodule structure by pre/post composition. The following are equivalent.

(1) For every finite-dimensional faithful left S-module V, the map $\rho : S \to \mathrm{End}_k(V)$ admits an S-bimodule retraction.

(2) S is separable over k (equivalently, the multiplication $\mu_S : S^e \to S$ splits S^e-linearly).

(3) For every injective k-algebra map $i : S \hookrightarrow A$ with $\dim_k A < \infty$, the inclusion splits as S-bimodules.

(4) The same as (3) for every injective k-algebra map $i : S \hookrightarrow A$ (no finiteness on A).

Proof. (4) $\Rightarrow$ (3) is clear. Also (3) $\Rightarrow$ (1) by taking $A = \mathrm{End}_k(V)$.

(1) $\Rightarrow$ (2): apply (1) to $V = S$ (left regular). Let $L : S \to \mathrm{End}_k(S)$ be left multiplication and let $\sigma : \mathrm{End}_k(S) \to S$ be an S-bimodule retraction of L. By adjunction, σ corresponds to a right S-module map $\theta : S^* = \mathrm{Hom}_k(S, k) \to S$. Then

$$\mu_S \circ (\mathrm{id}_S \otimes \theta) \circ L = \sigma \circ L = \mathrm{id}_S,$$

where $\mu_S : S \otimes_k S \to S$ is multiplication. Hence μ_S splits S-bilinearly (equivalently S^e-linearly), i.e. S is separable over k.

(2) $\Rightarrow$ (4): if S is separable over k and finite-dimensional, then S^e is semisimple, so every monomorphism of S^e-modules splits. □

Proposition 1. Let k be a field and $(R, \mathfrak{m})$ a finite-dimensional commutative local k-algebra. Assume that every nilpotent R-linear endomorphism of $\mathfrak{m}$ is given by multiplication by some element of R. Then every R-linear endomorphism of $\mathfrak{m}$ is given by multiplication by an element of R.

Proof. If $\mathfrak{m} = 0$, there is nothing to prove. Hence assume $\mathfrak{m} \neq 0$. Let $E := \mathrm{End}_R(\mathfrak{m})$ and let $\lambda : R \to E$ represent the R-action on $\mathfrak{m}$. If $\mathfrak{m} = I \oplus J$ with $I, J \neq 0$, pick $0 \neq f : I \to J$ and set $\psi|_I = f$, $\psi|_J = 0$; then $\psi^2 = 0$ but $\psi \notin \lambda(R)$, contradiction. Hence $\mathfrak{m}$ is indecomposable, so E is local (standard finite-length lemma), and $J(E)$ is nilpotent. $J(E) \subset \lambda(R)$ by hypothesis and it is easy to see $\lambda^{-1}(J(E)) = \mathfrak{m}$. Therefore it suffices to show that the map $\kappa := R/\mathfrak{m} \to D := E/J(E)$ is surjective. If $\mathfrak{m}^2 = 0$, then $E = \mathrm{End}_\kappa(\mathfrak{m})$ and the hypothesis forces $\dim_\kappa(\mathfrak{m}) = 1$, so $E = \lambda(R)$. Otherwise choose $t \geq 2$ with $\mathfrak{m}^{t+1} = 0 \neq \mathfrak{m}^t$ and set $V := \mathfrak{m}/\mathfrak{m}^2$, $G := \mathfrak{m}^t \subset \mathfrak{m}^2$; then $\mathfrak{m}V = \mathfrak{m}G = 0$, so any κ-linear $\theta : V \to G$ is R-linear. If $D \neq \kappa$, choose such a θ that is not D-linear and put $\psi := \iota\theta\pi$ with $\pi : \mathfrak{m} \twoheadrightarrow V$, $\iota : G \hookrightarrow \mathfrak{m}$; then $\psi^2 = 0$ since $\pi\iota = 0$, but ψ is not E-linear, so $\psi \notin \lambda(R)$, contradiction. Thus $D = \kappa$.

Corollary 1. Let k be a field and R a finite-dimensional commutative k-algebra. Then R is self-injective if and only if every finite injective ring map $R \hookrightarrow S$ splits as a map of R-modules.

Proof. The forward implication is obvious. Conversely, assume the splitting property; it is easy to reduce (via the Artinian product decomposition) to the case that $(R, \mathfrak{m})$ is local with residue field κ. By Proposition 1, it suffices to show that every nilpotent $\varphi \in \mathrm{End}_R(\mathfrak{m})$ is multiplication by an element of R (then $\mathrm{Ext}^1_R(\kappa, R) = 0$, hence R is self-injective). Choose $n \geq 1$ with $\varphi^n = 0$ and set

$$S := R[y]/(xy - \varphi(x)\ (x \in \mathfrak{m}),\ y^n).$$

Then $R \hookrightarrow S$ is a finite injection; choose an R-linear retraction $\pi : S \to R$ and put $a := \pi(\bar{y})$. Applying π to $x\bar{y} = \varphi(x)$ gives $xa = \varphi(x)$ for all $x \in \mathfrak{m}$, so φ is multiplication by a. $\square$